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Tip-loaded Cantilever Beams

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TIP-LOADED CANTILEVER BEAMS

... New graphs and table aid in determing vibration frequencies for a uniform cantilever beam having a mass at the end which is offset and which has rotatory inertia.

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In the vibration analysis of instruments and similar devices it is occasionally necessary to determine the natural frequencies of systems consisting of a uniform cantilever beam with a tip mass. If the mass may be considered to be a point mass concentrated at the tip of the cantilever, the problem is not difficult and the lowest frequency may be obtained with very great accuracy by using the Rayleigh approximation, that is by adding 23 per cent of the distributed mass to the tip mass and considering the problem as having but one degree of freedom. The resulting circular frequency is

$$\omega = \sqrt{3EI/L^3(M_{tip} + .23M_{beam})} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

However, higher frequencies are still hard to find, and, if there is any overhang (see Figure 1) between the end of the beam and the centroid of the tip mass, or if the tip mass possesses appreciable mass moment of inertia, even this Rayleigh approximation may be significantly in error.

Theoretical analysis of this case shows that the circular frequencies may be obtained as

$$\omega_i = u_i^2 \sqrt{EI/M_\beta L^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where u_i are the roots of the transcendental equation

$$ug_1(u)=R-\left(\frac{2B}{L}\right)u^2g_2(u)-\left(\frac{B^2+K^2}{L^2}\right)u^3g_3(u)+\left(\frac{1}{R}\right)\left(\frac{K}{L}\right)^2u^4g_4(u) \quad ..(3)$$

It should be explicitly noted that although the rotatory inertia of the tip mass is included in this formula, that of the beam-distributed mass is not. Also, the shearing deformations of the beam are not included, and the tip mass is assumed to be perfectly rigid. These simplifications are of utterly negligible significance except for quite unusual cases or for rather high modes.

In these formulas (see Figure 1)

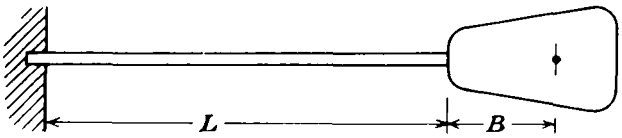

Figure 1.

$E=$ Young's modulus of beam material (psi)
$I=$ moment of inertia of beam cross section (in^4)
$L=$ Length of beam (in)
$M_\beta=$ mass of beam (lb. sec^2 in^{-1})
$=$ pounds weight of beam $\div$ 386
$R=$ weight of beam $\div$ weight of tip mass
$B=$ offset distance (in)
$K=$ centroidal radius of gyration of tip mass (in)

The functions $g_i(x)$ are:

$g_1(x)=(\sin x \cosh x-\cos x \sinh x)/(1+\cos x \cosh x)$ .....(4)

$g_2(x)=\sin x \sinh x/(1+\cos x \cosh x)$ .....(5)

$g_3(x)=(\sin x \cosh x+\cos x \sinh x)/(1+\cos x \cosh x)$ .....(6)

$g_4(x)=(1-\cos x \cosh x)/(1+\cos x \cosh x)$ .....(7)

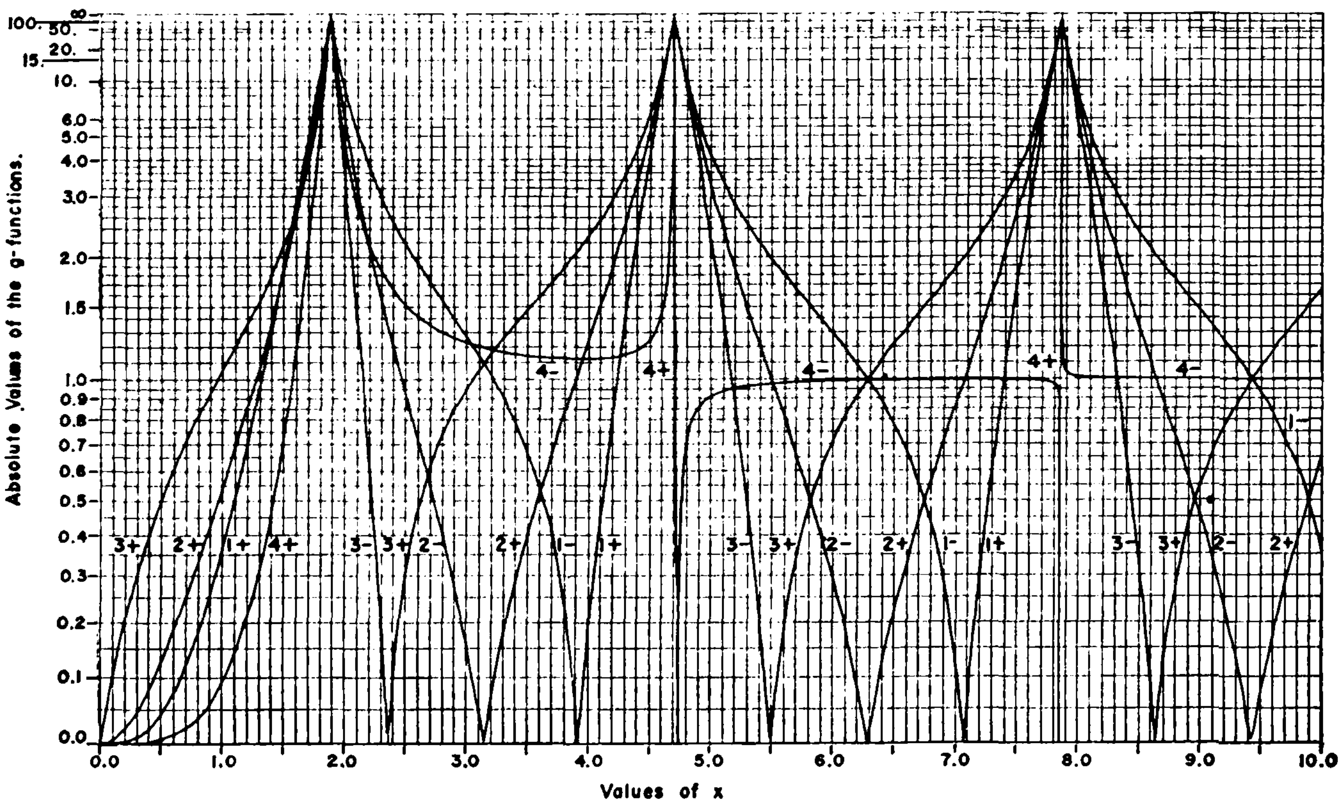

Figure 2. The numbers beside the curves indicate which g-function is represented. The plus or minus sign indicates the algebraic sign of the function in the region indicated.

	0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0	3.3334-9	5.3333-8	2.7000-7	8.5333-7	2.0833-6	4.3200-6	8.0033-6	1.3653-5	2.1870-5
.1	3.3334-5	4.8804-5	6.9121-5	9.5205-5	1.2806-4	1.6876-4	2.1846-4	2.7842-4	3.4995-4	4.3445-4
.2	5.3340-4	6.4837-4	7.8100-4	9.3301-4	1.1062-3	1.3025-3	1.5238-3	1.7722-3	2.0498-3	2.3589-3
.3	2.7017-3	3.0806-3	3.4981-3	3.9568-3	4.4591-3	5.0080-3	5.6061-3	6.2564-3	6.9619-3	7.7255-3
.4	8.5505-3	9.4402-3	1.0398-2	1.1427-2	1.2531-2	1.3713-2	1.4978-2	1.6328-2	1.7769-2	1.9303-2
.5	2.0936-2	2.2671-2	2.4513-2	2.6466-2	2.8534-2	3.0723-2	3.3037-2	3.5481-2	3.8060-2	4.0780-2
.6	4.3645-2	4.6661-2	4.9833-2	5.3168-2	5.6671-2	6.0349-2	6.4207-2	6.8252-2	7.2490-2	7.6928-2
.7	8.1573-2	8.6432-2	9.1513-2	9.6823-2	1.0237-1	1.0816-1	1.1420-1	1.2051-1	1.2708-1	1.3394-1
.8	1.4108-1	1.4852-1	1.5626-1	1.6433-1	1.7272-1	1.8145-1	1.9054-1	1.9998-1	2.0980-1	2.2000-1
.9	2.3061-1	2.4162-1	2.5307-1	2.6495-1	2.7729-1	2.9010-1	3.0340-1	3.1721-1	3.3153-1	3.4640-1
1.0	3.6183-1	3.7783-1	3.9444-1	4.1167-1	4.2954-1	4.4808-1	4.6731-1	4.8727-1	5.0796-1	5.2944-1
1.1	5.5171-1	5.7482-1	5.9880-1	6.2369-1	6.4951-1	6.7631-1	7.0413-1	7.3302-1	7.6300-1	7.9415-1
1.2	8.2650-1	8.6010-1	8.9502-1	9.3132-1	9.6906-1	1.0083 0	1.0491 0	1.0916 0	1.1358 0	1.1819 0
1.3	1.2298 0	1.2798 0	1.3319 0	1.3863 0	1.4430 0	1.5022 0	1.5641 0	1.6287 0	1.6963 0	1.7671 0
1.4	1.8412 0	1.9189 0	2.0003 0	2.0858 0	2.1756 0	2.2700 0	2.3693 0	2.4740 0	2.5843 0	2.7008 0
1.5	2.8239 0	2.9542 0	3.0923 0	3.2388 0	3.3945 0	3.5603 0	3.7370 0	3.9258 0	4.1279 0	4.3447 0
1.6	4.5777 0	4.8280 0	5.1002 0	5.3942 0	5.7139 0	6.0625 0	6.4441 0	6.8635 0	7.3266 0	7.8404 0
1.7	8.4136 0	9.0569 0	9.7839 0	1.0612 1	1.1563 1	1.2668 1	1.3965 1	1.5510 1	1.7381 1	1.9693 1
1.8	2.2622 1	2.6453 1	3.1675 1	3.9215 1	5.1054 1	7.2327 1	1.2178 2	3.6500 2	-3.8537 2	-1.2826 2
1.9	-7.7704 1	-5.6124 1	-4.4159 1	-3.6555 1	-3.1295 1	-2.7440 1	-2.4494 1	-2.2170 1	-2.0289 1	-1.8737 1
2.0	-1.7433 1	-1.6323 1	-1.5366 1	-1.4534 1	-1.3802 1	-1.3154 1	-1.2577 1	-1.2059 1	-1.1592 1	-1.1168 1
2.1	-1.0782 1	-1.0429 1	-1.0105 1	-9.8062 0	-9.5302 0	-9.2742 0	-9.0363 0	-8.8145 0	-8.6072 0	-8.4131 0
2.2	-8.2309 0	-3.0595 0	-7.8981 0	-7.7456 0	-7.6015 0	-7.4650 0	-7.3355 0	-7.2125 0	-7.0955 0	-6.9840 0
2.3	-6.8777 0	-6.7761 0	-6.6790 0	-6.5861 0	-6.4970 0	-6.4115 0	-6.3295 0	-6.2506 0	-6.1747 0	-6.1016 0
2.4	-6.0311 0	-5.9631 0	-5.8974 0	-5.8339 0	-5.7725 0	-5.7131 0	-5.6555 0	-5.5997 0	-5.5455 0	-5.4929 0
2.5	-5.4417 0	-5.3920 0	-5.3436 0	-5.2965 0	-5.2506 0	-5.2058 0	-5.1621 0	-5.1195 0	-5.0778 0	-5.0370 0
2.6	-4.9972 0	-4.9581 0	-4.9199 0	-4.8824 0	-4.8457 0	-4.8097 0	-4.7743 0	-4.7395 0	-4.7053 0	-4.6717 0
2.7	-4.6386 0	-4.6061 0	-4.5740 0	-4.5423 0	-4.5111 0	-4.4804 0	-4.4500 0	-4.4200 0	-4.3903 0	-4.3610 0
2.8	-4.3320 0	-4.3032 0	-4.2748 0	-4.2466 0	-4.2187 0	-4.1909 0	-4.1634 0	-4.1361 0	-4.1090 0	-4.0821 0
2.9	-4.0553 0	-4.0287 0	-4.0022 0	-3.9758 0	-3.9495 0	-3.9233 0	-3.8972 0	-3.8712 0	-3.8452 0	-3.8193 0
3.0	-3.7934 0	-3.7676 0	-3.7417 0	-3.7159 0	-3.6901 0	-3.6642 0	-3.6384 0	-3.6125 0	-3.5866 0	-3.5606 0
3.1	-3.5345 0	-3.5084 0	-3.4822 0	-3.4560 0	-3.4296 0	-3.4031 0	-3.3765 0	-3.3498 0	-3.3229 0	-3.2959 0
3.2	-3.2688 0	-3.2414 0	-3.2140 0	-3.1863 0	-3.1584 0	-3.1304 0	-3.1021 0	-3.0736 0	-3.0449 0	-3.0159 0
3.3	-2.9867 0	-2.9572 0	-2.9275 0	-2.8974 0	-2.8671 0	-2.8365 0	-2.8056 0	-2.7743 0	-2.7427 0	-2.7108 0
3.4	-2.6785 0	-2.6458 0	-2.6127 0	-2.5793 0	-2.5454 0	-2.5111 0	-2.4764 0	-2.4412 0	-2.4056 0	-2.3695 0
3.5	-2.3329 0	-2.2958 0	-2.2581 0	-2.2199 0	-2.1812 0	-2.1419 0	-2.1020 0	-2.0615 0	-2.0203 0	-1.9785 0
3.6	-1.9360 0	-1.8929 0	-1.8490 0	-1.8044 0	-1.7590 0	-1.7129 0	-1.6660 0	-1.6182 0	-1.5695 0	-1.5200 0
3.7	-1.4696 0	-1.4182 0	-1.3659 0	-1.3125 0	-1.2581 0	-1.2027 0	-1.1461 0	-1.0884 0	-1.0295 0	-9.6941-1
3.8	-9.0802-1	-8.4533-1	-7.8128-1	-7.1582-1	-6.4891-1	-5.8049-1	-5.1051-1	-4.3891-1	-3.6563-1	-2.9060-1
3.9	-2.1377-1	-1.3505-1	-5.4372-2	2.8334-2	1.1315-1	2.0017-1	2.8948-1	3.8117-1	4.7535-1	5.7212-1
4.0	6.7159-1	7.7389-1	8.7913-1	9.8746-1	1.0990 0	1.2139 0	1.3324 0	1.4545 0	1.5806 0	1.7107 0
4.1	1.8451 0	1.9840 0	2.1276 0	2.2762 0	2.4301 0	2.5895 0	2.7548 0	2.9262 0	3.1042 0	3.2892 0
4.2	3.4815 0	3.6816 0	3.8900 0	4.1073 0	4.3340 0	4.5708 0	4.8183 0	5.0774 0	5.3487 0	5.6334 0
4.3	5.9323 0	6.2466 0	6.5775 0	6.9264 0	7.2948 0	7.6845 0	8.0972 0	8.5351 0	9.0007 0	9.4967 0
4.4	1.0026 1	1.0592 1	1.1200 1	1.1853 1	1.2557 1	1.3319 1	1.4145 1	1.5045 1	1.6028 1	1.7107 1
4.5	1.8297 1	1.9616 1	2.1086 1	2.2734 1	2.4596 1	2.6715 1	2.9150 1	3.1976 1	3.5297 1	3.9255 1
4.6	4.4054 1	4.9992 1	5.7532 1	6.7423 1	8.0969 1	1.0066 2	1.3189 2	1.8905 2	3.2733 2	1.1416-3
4.7	-8.0019 2	-3.0083 2	-1.8694 2	-1.3648 2	-1.0800 2	-8.9706 1	-7.6961 1	-6.7572 1	-6.0367 1	-5.4663 1
4.8	-5.0035 1	-4.6205 1	-4.2981 1	-4.0231 1	-3.7857 1	-3.5787 1	-3.3965 1	-3.2349 1	-3.0907 1	-2.9610 1
4.9	-2.8439 1	-2.7376 1	-2.6406 1	-2.5518 1	-2.4701 1	-2.3947 1	-2.3249 1	-2.2602 1	-2.1999 1	-2.1436 1
5.0	-2.0909 1	-2.0415 1	-1.9951 1	-1.9513 1	-1.9101 1	-1.8711 1	-1.8342 1	-1.7992 1	-1.7659 1	-1.7343 1
5.1	-1.7042 1	-1.6755 1	-1.6480 1	-1.6218 1	-1.5967 1	-1.5727 1	-1.5496 1	-1.5275 1	-1.5062 1	-1.4857 1
5.2	-1.4660 1	-1.4470 1	-1.4287 1	-1.4110 1	-1.3939 1	-1.3773 1	-1.3613 1	-1.3459 1	-1.3309 1	-1.3163 1
5.3	-1.3022 1	-1.2885 1	-1.2752 1	-1.2622 1	-1.2496 1	-1.2374 1	-1.2255 1	-1.2138 1	-1.2025 1	-1.1914 1
5.4	-1.1806 1	-1.1701 1	-1.1598 1	-1.1498 1	-1.1399 1	-1.1303 1	-1.1209 1	-1.1116 1	-1.1026 1	-1.0937 1
5.5	-1.0850 1	-1.0765 1	-1.0681 1	-1.0599 1	-1.0519 1	-1.0439 1	-1.0361 1	-1.0285 1	-1.0209 1	-1.0135 1
5.6	-1.0062 1	-9.9900 0	-9.9191 0	-9.8492 0	-9.7802 0	-9.7122 0	-9.6451 0	-9.5789 0	-9.5135 0	-9.4489 0
5.7	-9.3851 0	-9.3220 0	-9.2596 0	-9.1979 0	-9.1369 0	-9.0765 0	-9.0166 0	-8.9574 0	-8.8987 0	-8.8406 0
5.8	-8.7829 0	-8.7257 0	-8.6690 0	-8.6128 0	-8.5569 0	-8.5015 0	-8.4464 0	-8.3917 0	-8.3374 0	-8.2834 0
5.9	-8.2297 0	-8.1762 0	-8.1231 0	-8.0702 0	-8.0176 0	-7.9652 0	-7.9130 0	-7.8611 0	-7.8093 0	-7.7576 0
6.0	-7.7062 0	-7.6549 0	-7.6037 0	-7.5526 0	-7.5016 0	-7.4507 0	-7.3999 0	-7.3492 0	-7.2985 0	-7.2478 0
6.1	-7.1972 0	-7.1466 0	-7.0960 0	-7.0453 0	-6.9947 0	-6.9440 0	-6.8933 0	-6.8425 0	-6.7916 0	-6.7407 0
6.2	-6.6896 0	-6.6385 0	-6.5872 0	-6.5358 0	-6.4843 0	-6.4326 0	-6.3807 0	-6.3287 0	-6.2764 0	-6.2240 0
6.3	-6.1713 0	-6.1185 0	-6.0653 0	-6.0120 0	-5.9583 0	-5.9044 0	-5.8502 0	-5.7956 0	-5.7409 0	-5.6856 0
6.4	-5.6301 0	-5.5742 0	-5.5180 0	-5.4613 0	-5.4043 0	-5.3468 0	-5.2889 0	-5.2305 0	-5.1717 0	-5.1124 0
6.5	-5.0526 0	-4.9923 0	-4.9315 0	-4.8701 0	-4.8081 0	-4.7456 0	-4.6824 0	-4.6187 0	-4.5543 0	-4.4892 0
6.6	-4.4234 0	-4.3570 0	-4.2898 0	-4.2219 0	-4.1532 0	-4.0837 0	-4.0134 0	-3.9422 0	-3.8702 0	-3.7973 0
6.7	-3.7235 0	-3.6487 0	-3.5729 0	-3.4962 0	-3.4183 0	-3.3395 0	-3.2595 0	-3.1783 0	-3.0960 0	-3.0125 0
6.8	-2.9278 0	-2.8417 0	-2.7543 0	-2.6656 0	-2.5754 0	-2.4838 0	-2.3907 0	-2.2960 0	-2.1998 0	-2.1018 0
6.9	-2.0022 0	-1.9008 0	-1.7975 0	-1.6924 0	-1.5853 0	-1.4762 0	-1.3650 0	-1.2516 0	-1.1360 0	-1.0180 0
7.0	-8.9768-1	-7.7484-1	-6.4942-1	-5.2131-1	-3.9042-1	-2.5665-1	-1.1987-1	2.0020-2	1.6315-1	3.0964-1
7.1	4.5964-1	6.1329-1	7.7072-1	9.3212-1	1.0976 0	1.2674 0	1.4417 0	1.6207 0	1.8046 0	1.9936 0
7.2	2.1879 0	2.3878 0	2.5936 0	2.8056 0	3.0240 0	3.2492 0	3.4814 0	3.7212 0	3.9688 0	4.2247 0
7.3	4.4894 0	4.7632 0	5.0468 0	5.3407 0	5.6455 0	5.9618 0	6.2904 0	6.6319 0	6.9873 0	7.3574 0
7.4	7.7431 0	8.1456 0	8.5660 0	9.0055 0	9.4656 0	9.9477 0	1.0454 1	1.0985 1	1.1544 1	1.2133 1
7.5	1.2754 1	1.3411 1	1.4106 1	1.4843 1	1.5626 1	1.6459 1	1.7348 1	1.8299 1	1.9317 1	2.0412 1
7.6	2.1591 1	2.2866 1	2.4248 1	2.5751 1	2.7393 1	2.9195 1	3.1179 1	3.3377 1	3.5825 1	3.8568 1
7.7	4.1664 1	4.5186 1	4.9228 1	5.3915 1	5.9417 1	6.5966 1	7.3894 1	8.3690 1	9.6102 1	1.1234 2
7.8	1.3451 2	1.6658 2	2.1708 2	3.0838 2	5.2339 2	1.6422 3	-1.5071 3	-5.2414 2	-3.1998 2	-2.3167 2
7.9	-1.8239 2	-1.5095 2	-1.2913 2	-1.1312 2	-1.0085 2	-9.1163 1	-8.3310 1	-7.6816 1	-7.1357 1	-6.6703 1
8.0	-6.2687 1	-5.9186 1	-5.6107 1	-5.3377 1	-5.0939 1	-4.8750 1	-4.6772 1	-4.4976 1	-4.3338 1	-4.1838 1
8.1	-4.0458 1	-3.9185 1	-3.8007 1	-3.6913 1	-3.5894 1	-3.4942 1	-3.4052 1	-3.3217 1	-3.2432 1	-3.1693 1
8.2	-3.0995 1	-3.0335 1	-2.9711 1	-2.9118 1	-2.8555 1	-2.8020 1	-2.7510 1	-2.7023 1	-2.6558 1	-2.6114 1
8.3	-2.5688 1	-2.5280 1	-2.4889 1	-2.4513 1	-2.4152 1	-2.3804 1	-2.3469 1	-2.3146 1	-2.2835 1	-2.2534 1
8.4	-2.2243 1	-2.1962 1	-2.1690 1	-2.1427 1	-2.1171 1	-2.0923 1	-2.0683 1	-2.0450 1	-2.0223 1	-2.0002 1
8.5	-1.9787 1	-1.9579 1	-1.9375 1	-1.9177 1	-1.8983 1	-1.8795 1	-1.8611 1	-1.8431 1	-1.8255 1	-1.8083 1
8.6	-1.7915 1	-1.7751 1	-1.7590 1	-1.7432 1	-1.7278 1	-1.7127 1	-1.6978 1	-1.6833 1	-1.6690 1	-1.6549 1
8.7	-1.6412 1	-1.6276 1	-1.6143 1	-1.6012 1	-1.5884 1	-1.5757 1	-1.5632 1	-1.5509 1	-1.5388 1	-1.5269 1
8.8	-1.5152 1	-1.5036 1	-1.4922 1	-1.4809 1	-1.4697 1	-1.4587 1	-1.4479 1	-1.4371 1	-1.4265 1	-1.4161 1
8.9	-1.4057 1	-1.3954 1	-1.3853 1	-1.3752 1	-1.3653 1	-1.3554 1	-1.3457 1	-1.3360 1	-1.3264 1	-1.3169 1
9.0	-1.3074 1	-1.2981 1	-1.2888 1	-1.2796 1	-1.2704 1	-1.2613 1	-1.2523 1	-1.2433 1	-1.2344 1	-1.2255 1
9.1	-1.2167 1	-1.2079 1	-1.1992 1	-1.1905 1	-1.1818 1	-1.1732 1	-1.1646 1	-1.1561 1	-1.1475 1	-1.1391 1
9.2	-1.1306 1	-1.1221 1	-1.1137 1	-1.1053 1	-1.0969 1	-1.0885 1	-1.0802 1	-1.0718 1	-1.0635 1	-1.0552 1
9.3	-1.0468 1	-1.0385 1	-1.0302 1	-1.0219 1	-1.0136 1	-1.0052 1	-9.9689 0	-9.8855 0	-9.8020 0	-9.7183 0
9.4	-9.6346 0	-9.5506 0	-9.4665 0	-9.3823 0	-9.2978 0	-9.2131 0	-9.1281 0	-9.0429 0	-8.9573 0	-8.8715 0
9.5	-8.7854 0	-8.6989 0	-8.6120 0	-8.5248 0	-8.4371 0	-8.3490 0	-8.2605 0	-8.1715 0	-8.0820 0	-7.9921 0
9.6	-7.9015 0	-7.8104 0	-7.7188 0	-7.6265 0	-7.5336 0	-7.4401 0	-7.3459 0	-7.2509 0	-7.1553 0	-7.0589 0
9.7	-6.9617 0	-6.8637 0	-6.7649 0	-6.6652 0	-6.5646 0	-6.4631 0	-6.3607 0	-6.2572 0	-6.1528 0	-6.0473 0
9.8	-5.9407 0	-5.8329 0	-5.7240 0	-5.6140 0	-5.5026 0	-5.3900 0	-5.2761 0	-5.1609 0	-5.0442 0	-4.9261 0
9.9	-4.8065 0	-4.6853 0	-4.5626 0	-4.4382 0	-4.3121 0	-4.1843 0	-4.0546 0	-3.9231 0	-3.7897 0	-3.6543 0
10.0	-3.5168 0	-3.3772 0	-3.2354 0	-3.0913 0	-2.9448 0	-2.7960 0	-2.6446 0	-2.4907 0	-2.3340 0	-2.1746 0

TABLE 1—Values of the product $ug_1(u)$. The entries in the body of the table consist of a mantissa followed by the exponent of a multiplicative power of ten. Thus, the entry corresponding to $u=.67$ is .068252 while that corresponding to $u=4.73$ is −136.48.

Graphs of the functions $g_1(x)$ through $g_4(x)$ are shown in Figure 2 for values of x up to 10.0. Note that the vertical scale runs all the way from zero to infinity and that only the numerical values have been plotted. The correct algebraic sign can be obtained as described. For example, $g_2(2.5) = -.93$ (approximately). A tabulation of the product $ug_1(u)$ is given in Table I. This permits immediate solution, for all roots, if the tip mass is concentrated at the tip of the beam so that $B=K=0$. Thus, for example, if $R=.6$, we have $u_1=1.1205$; $u_2=3.9928$; $u_3=7.1091$, determined by interpolation. The fourth and higher roots lie outside the range of the table, but a way will presently be indicated for finding such roots.

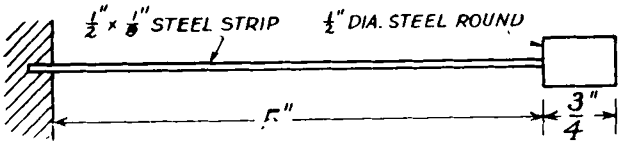

Figure 3.

To indicate a method of solution for more complicated cases, consider the case illustrated in Figure 3. In this case

$$E = 30{,}000{,}000 \text{ psi}$$
$$R = [(5)(1/2)(1/8)]/\ [(\pi)(1/4)^2(3/4)] = 2.122$$
$$M_\beta = (5)(1/2)(1/8)(.283)/(386) = .0002291 \text{ lb.sec.}^2 \text{ in.}^{-1}$$
$$I = (1/2)(1/8)^3/(12) = .00008138 \text{ in.}^4$$
$$\sqrt{EI/M_\beta L^3} = 292.0 \text{ sec.}^{-1}$$
$$2B/L = (2)(3/8)/(5) = .15$$
$$(K^2+B^2)/L^2 = [3(1/4)^2+4(3/4)^2]/(12)(25) = .008125$$
$$(K/L)^2 = .008125 - (B/L)^2 = .008125 - (.075)^2 = .0025$$
$$(1/R)(K/L)^2 = .00118$$

and the equation becomes

$$ug_1(u) = 2.122 - .15\ u^2g_2(u) - .008125\ u^3g_3(u) + .00118\ u^4g_4(u) = Q$$

In this form, Q denotes all the quantities on the right. Since we do not yet know u, we take Q initially as 2.122. From Table I, we find $u=1.43$ (for the first mode). Next, we reevaluate

$$Q = 2.122 - (.15)(1.43)^2(1.38) - (.008125)(1.43)^3(1.9) + (.00118)(1.43)^4(.54) = 1.657$$

Here the values $g_2(1.43) = 1.38$, and so on, are read from the graph of Figure 2. Corresponding to $Q=1.657$, we find $u=1.38$. Feeding this back in, we get $Q=1.719$ from which $u=1.383$ (by interpolation). There is no corresponding change in Q (to the accuracy permitted in reading Figure 2), so that we have $u=1.383$, $\omega_1=292(1.383)^2=558.5$ radians per second. (A more precise result than can be obtained from Figure 2, gives $u=1.38215$.)

We can also find the second mode, proceeding in much the same way. Corresponding to $Q=2.122$, we get $u=4.12$. Reevaluating, $Q=-3.87$, $u=2.97$, say 3. Reevaluating, $Q=2.00$, $u=4.1$. Clearly this is converging slowly if at all. The correct result lies between 3 and 4.1. Arbitrarily selecting 3.5 gives $Q=.688$, $u=4$. This is still too high. Using $u=3.7$ gives $u=3.9$. Using $u=3.8$ gives $u=3.8$, approximately. Trying more carefully with $u=3.8$, we get $u=3.813$, $\omega_2=4240$ radians per second. (A more precise result obtained by a different analysis gives $u=3.8113$.)

An inspection of Figure 2 shows that except for small values of u, the shape of the curves repeats in an interesting and useful manner, reflecting about the values $x=7.86$, $x=9.42$, etc. These values are very nearly 2.5π, 3π, 3.5π, etc. This knowledge permits finding solutions beyond the scope of Figure 2. Thus, for the first problem $R=.6$; $B=K=0$, we got $u_1=1.1205$, $u_2=3.9928$, $u_3=7.1091$. Let us now get u_4, which is in the neighborhood $+7.1+(7.1-4.0)=10.2$. Thus, $g_1(u_4)$ is approximately $(.6)/(10.2)=.06$. Since $g_1(3\pi+y) = -g_3(3\pi-y)$, we will look for $-g_3(8.64)$ which should be approximately .06. Actually we find that $-g_3(8.61)=.06$. Thus $u_4=6\pi-8.61=10.24$. Actually we could have gotten a good value by taking $u_4=u_3+(u_3-u_2)=10.22$. Using the slightly better result, we have $u_5=2u_4-u_3=13.37$, and so on.